Final Practice
Math 181B, UCSD, Spring 2018

Exercise 1

We consider the linear model

$$Y_i = a + bW_i + cZ_i + \varepsilon_i,$$

where the W_i's, and Z_i's are fixed real numbers, with $\varepsilon_1, \dots, \varepsilon_n \sim_{iid} N(0, \sigma^2)$, and a, b, c, σ^2 are unknown real parameters. We denote by W, Y, Z the column vectors of size n with respective components $(W_i)_i, (Y_i)_i, (Z_i)_i$. The mean is $\bar{W} = \frac{1}{n}\sum_{i=1}^n W_i$, the dot product $W'Z = \sum_{i=1}^n W_i Z_i$, the norm squared $\|W\|^2 = W'W$, and similarly for the other variables.
We assume that $\bar{W} = \bar{Z} = 0$, and that

$$\|W\| = \|Z\| = r > 0, \quad W'Z = r^2 \sin\theta \text{ for } \theta \in (-\pi/2, \pi/2).$$

1. Write the least squares estimators $\hat{a}, \hat{b}, \hat{c}$ of a, b, c as functions of r, θ, and dot products of above form. Give the joint distribution of $(\hat{a}, \hat{b}, \hat{c}, \hat{\sigma}^2)$, where $\hat{\sigma}^2$ is the usual unbiased estimator of σ^2.

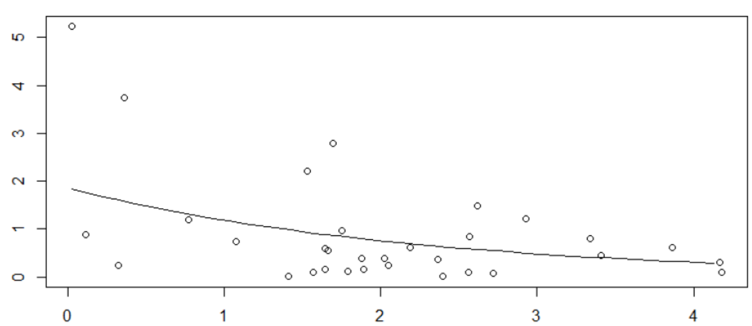
2. Give a confidence interval of level $1-\alpha$ for c. What is the expected value of the square of its length? How does it vary with θ?

3. Give a confidence cuboid for (a, b, c) with level 97% when $n = 27$.

4. Build a confidence ellipsoid for (a, b, c) with level 97% when $n = 27$.

Exercise 2

The figure below represent lifetime data of 32 components of engine pieces, as a function of their corrosion level x. We describe the distribution of lifetime of components by an exponential distri-



bution of which the mean depends on the corrosion index x_i: $Y_i \sim \mathcal{E}(\lambda_i)$ with

$$\mathbb{E}[Y_i] = \frac{1}{\lambda_i} = \exp(\beta_1 + \beta_2 x_i),$$

for $i \in \{1, \dots, 32\}$.

1. Justify this modeling choice.

2. Write down the likelihood of the observations.

3. Write the equations defining the maximum likelihood estimator $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)$. How would you solve it? (You are not asked to actually solve them)

4. Compute the Fisher information matrix of the sample. Are we in a standard setting of maximum likelihood estimation?

We will admit that the MLE is asymptotically Gaussian if there exists two constants a_1 and a_2 such that $\lim_n \sum_i x_i/n = a_1$ and $\lim_i \sum_i x_i^2/n = a_2$ and $a_1^2 \neq a_2$.

5. Build and sketch a confidence region of level 95% for $\beta = (\beta_1, \beta_2)$

(a) which is a rectangle; (Bonferroni-type)

(b) which is an ellipse. (Wald-type)

6. Test H_0: $\beta_2 = 0$ against H_1: $\beta_2 \neq 0$

(a) using a Wald test;

(b) using a likelihood ratio test.